(I) Contractibility of the Arc Complex

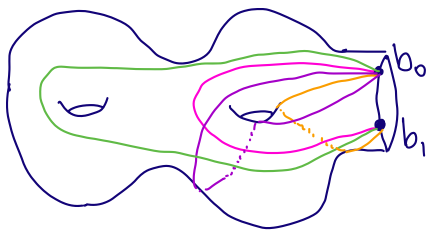
$S_{g,r}$: compact oriented surface w/ genus g, r boundary components

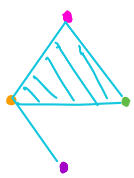
Fix two points b_0, b_1 on the boundary (may or may not lie on same bdry component)

The Disordered Arc Complex is a subcplex of the full arc complex $\mathcal{A}(S_{g,r}, \{b_0, b_1\})$

vertices $\leftrightarrow$ isotopy classes of arcs (no self-intersections) w/ endpts in $\{b_0, b_1\}$

p-simplices $\leftrightarrow$ $(p+1)$ arcs that can be realised pairwise disjointly (except at endpts)] "arc systems"





$\boxed{\text{Thm: } A(S, \{b_0, b_1\}) \text{ is contractible}}$

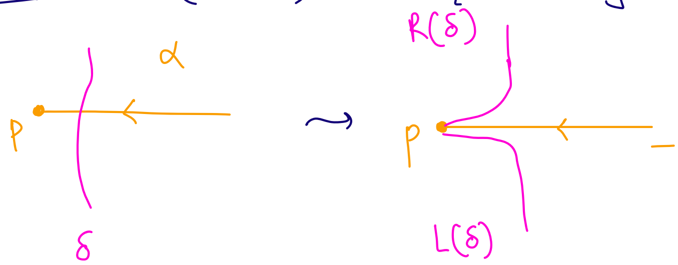
Fix an arc α in $A(S, \{b_0, b_1\})$ and an orientation on it.

Will construct a retraction of the complex onto $\mathrm{Star}(\alpha)$

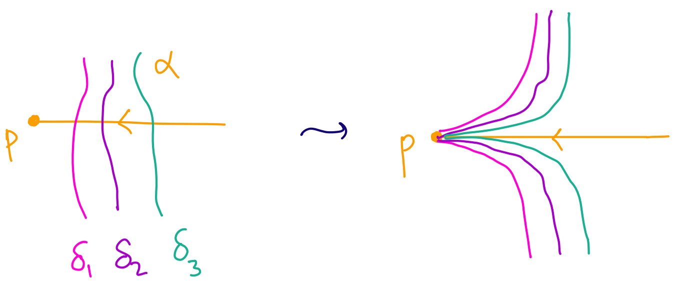
$= \{\sigma \mid \alpha \cup \sigma \in A(S)\}$

Is a cone w/ cone pt α

Note : $\mathrm{Vertices}(\mathrm{Star}(\alpha)) = \alpha \cup \{\text{arcs disjoint from } \alpha\}$



Atmost one of $L(\delta), R(\delta)$ is trivial



(II) The Disordered Arc Complex

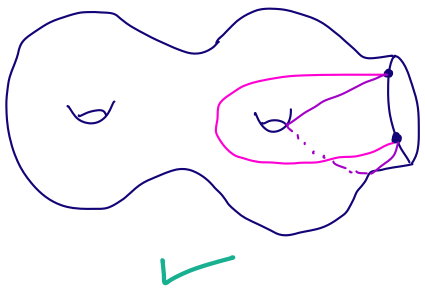
Fix $S_{g,r}$; $b_0, b_1 \in \partial S_{g,r}$

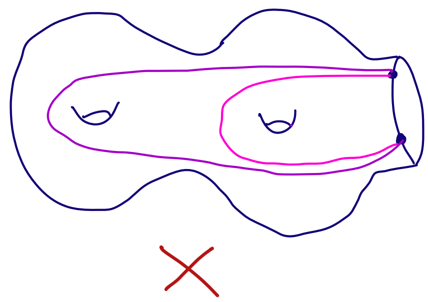
$\mathcal{D}^{\nu}(S_{g,r}; b_0, b_1)$: arc systems $\langle \alpha_0, \alpha_1, \dots, \alpha_p \rangle$ s.t. :

$\nu = \begin{cases} 1 & \text{if } b_0, b_1 \text{ on same bdry component} \\ 2 & \text{o/w} \end{cases}$

1. The endpts of α_i are b_0 & b_1
2. Cutting along $\alpha_0, \alpha_1, \dots, \alpha_p$ doesn't disconnect S
3. The order of $\alpha_0, \alpha_1, \dots, \alpha_p$ at b_0 and b_1 is the same

(ensures $\mathrm{Mod}(S_{g,r})$ action is transitive on p-simplices)





$$\mathcal{D}^{\nu}(S_{g,r}; b_0, b_1) \subset \underbrace{B_0(S_{g,r}; b_0, b_1)}_{(2g+\nu-3)\text{-connected}}^{\text{coconnected arc systems}} \subset B(S_{g,r}; b_0, b_1) \subset A(S_{g,r}; \{b_0, b_1\}) \simeq *$$

$$\boxed{\textbf{Thm}: \ \mathcal{D}^{\nu}(S_{g,r}; b_0, b_1) \text{ is } \left(\frac{2g+\nu-5}{3}\right)\text{-connected}}$$

We shall prove this assuming the connectivity of $B_0(S_{g,r}; b_0, b_1)$

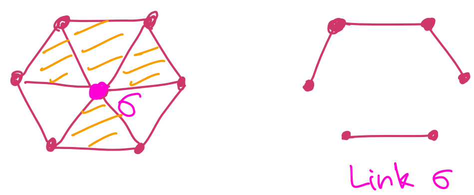
Rmk: (1) Connectivity of $B_0(S_{g,r}; b_0, b_1)$ can be proven using contractibility of $A(S_{g,r}; \{b_0, b_1\})$ using a similar idea.

(2) Note the result is true when $g = 0$.

Defn: For X a simplicial complex, σ : simplex,

$\mathrm{Link}_X \sigma := \{\tau \mid \sigma \cup \tau \in X, \ \sigma \cap \tau = \phi\}$

(Thus $\mathrm{star}_X \sigma = \sigma * \mathrm{Link}_X \sigma$)



III Proof Strategy : "Badness" Arguments

Note: When $g \geq 1$, $\frac{2g+\nu-5}{3} < 2g+\nu-3$

So for $k \leq \frac{2g+\nu-5}{3}$, given a map $f : S^k \to \mathcal{D}^{\nu}(S_{g,r}; b_0, b_1)$, we can extend it :

$$\begin{tikzcd} S^k \arrow[r, "f"] \arrow[d, hook] & \mathcal{D}^{\nu}(S_{g,r}; b_0, b_1) \arrow[d, hook] \\ D^{k+1} \arrow[r, "\hat{f}"] \arrow[ur, dashed] & \mathcal{B}_0(S_{g,r}; b_0, b_1) \end{tikzcd}$$

Aim: Modify $\hat{f}$ until dotted map exists.

Here's the broad idea :

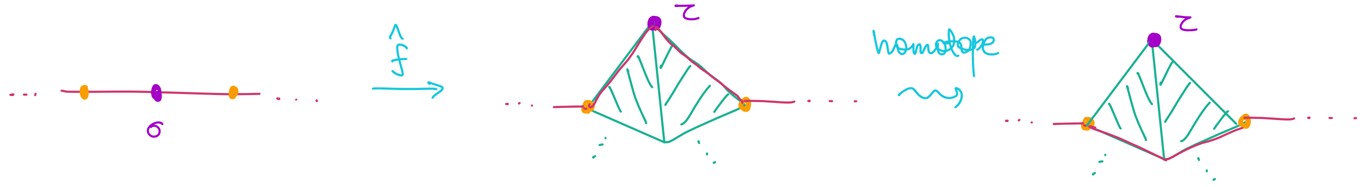
The image of $\hat{f}$ consists of "good" and "bad" simplices

"good" $\downarrow$ $\in \mathcal{D}(S; b_0, b_1)$

"bad" $\downarrow$ $\notin \mathcal{D}(S; b_0, b_1)$

We want to "push" $\hat{f}$ off all the bad simplices.

Eg:



Here's why we're able to do this in the above example :

- The link of σ is $\cong S^0$, and maps to the link of τ. Moreover, it maps only to good simplices in $\mathrm{lk}\,\tau$.
- Since lk_{τ} is 0-connected, we're able to fill this in with a 1-ball.
- $f|_{\mathrm{star}(\sigma)}$ and this 1-ball now bound a 2-ball, using which we're able to homotope $\hat{f}$ off of τ.

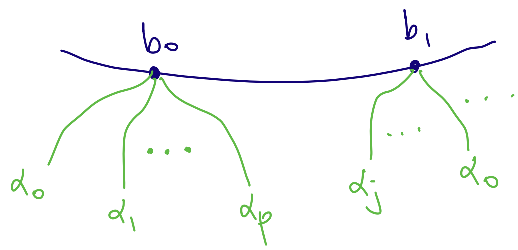
Tools to do this:

① Assume $\hat{f} : D^{k+1} \to B_0(S_{g,r}; b_0, b_1)$ is simplicial.

PL Topology $\rightsquigarrow$ can assume that simplicial structure on D^{k+1} is s.t. $lk_{D^{k+1}}\, \sigma \cong S^{k-p}$, where $p = \dim \sigma$, $\sigma \notin \partial D^{k+1}$

② For a simplex τ in $B_0(S_{g,r}; b_0, b_1)$,

- τ is "good" if $\tau \in D(S_{g,r}; b_0, b_1)$
- τ is "bad" if $\tau = \langle \alpha_0, \alpha_1, \ldots, \alpha_p \rangle$ and:



Note: Any simplex not in $D(S_{g,r}; b_0, b_1)$ contains a bad simplex as a face.
So it'll be enough to homotope $\hat{f}$ off all the bad simplices.

③ Let $\sigma \in D^{k+1}$ be of maximal dimension s.t. $\tau = \hat{f}(\sigma)$ is bad.

Then, maximality $\Rightarrow$ $\hat{f}(lk\, \sigma) \subset lk(\tau)$

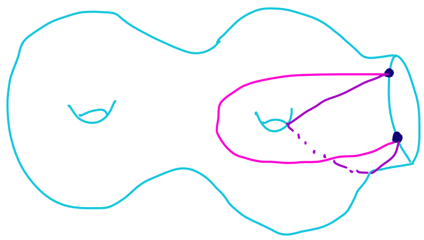
$\hat{f}(lk(\sigma))$ is good

If we show: $\hat{f}(lk(\sigma))$ is contained in a $(k-p)$-connected good subcomplex of $lk(\tau)$,
we'll be able to fill $\hat{f}_{|lk(\sigma)}$ with a ball,
and homotope $\hat{f}$ as explained above

We will show that $lk(\tau) \cap D(S_{g,r}, b_0, b_1) \cong D(S_{g',r'}, b_0', b_1')$ where $g' < g$, and deduce $(k-p)$-connectivity inductively.

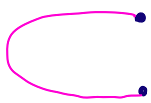
(IV) Analysing Links

Simplices in link of $\langle \alpha_0, \alpha_1, \dots, \alpha_p \rangle$



$\rightsquigarrow$ how many more arcs can we add to this system?

To understand this, we need to understand the resultant surface S' after we cut along $\alpha_0, \alpha_1, \dots, \alpha_p$.



$\chi = 1$

Euler characteristic argument : With each cut, χ goes up by 1.

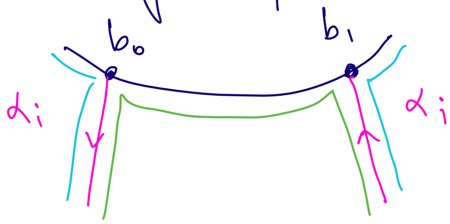
$$\chi = 2 - 2g - r$$

Lemma : If we cut $\alpha_0, \alpha_1, \dots, \alpha_p$ in that order, then at every cut, either $\begin{matrix} r \mapsto r+1 \\ g \mapsto g-1 \end{matrix}$ or $\begin{matrix} r \mapsto r-1 \\ g \mapsto g \end{matrix}$

In particular, g decreases every other time.

Case 1

b_0, b_1 on same bdry component

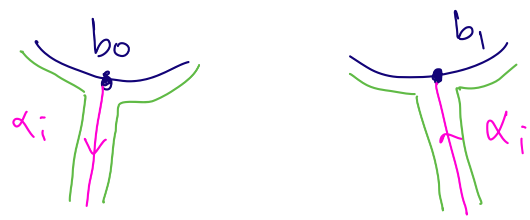


- "inner part
- "outer part

$r \mapsto r+1$
$g \mapsto g-1$

Case 2

b_0, b_1 on different bdry components

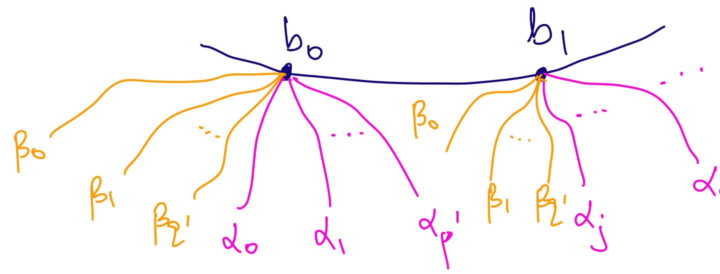


$r \mapsto r-1$
$g \mapsto g$

Back to badness:

Suppose $\sigma = [v_0, v_1, \ldots, v_p]$ is maximal s.t.
$\tau = \hat{f}(\sigma) = \langle \alpha_0, \alpha_1, \ldots, \alpha_{p'} \rangle$ is bad.

Suppose $\gamma = [w_0, w_1, \ldots, w_q] \in \mathrm{lk}\, \sigma$.
Let $\langle \beta_0, \beta_1, \ldots, \beta_{q'} \rangle = \hat{f}(\gamma)$



Then, maximality of σ implies:

- $\beta_i < \alpha_0$ at b_0 $\forall i$
- $\beta_i < \alpha_j$ at b_1 $\forall i, j$
- $\beta_i < \beta_j$ at $b_0 \iff \beta_i < \beta_j$ at b_1

This implies that $\langle \beta_0, \beta_1, \ldots, \beta_{q'} \rangle$ can be identified with a disordered arc system on the surface
$S' = S \setminus \langle \alpha_0, \ldots, \alpha_{p'} \rangle$.

cut along
$\alpha_0, \ldots, \alpha_{p'}$

By our Lemma, the genus g' of S' is $\geq g - p - 1$.

Thus $D(S'; b_0', b_1')$ is $\left(\frac{2(g-p-1)-4}{3}\right)$-connected.

Note: $k - p \leq \frac{2g+\nu-5}{3} - p = \frac{2g-3p-5+\nu}{3} = \frac{2(g-p-1)-p-3+\nu}{3}$

$\leq \frac{2(g-p-1)-4}{3}$ when $p \geq 2$, $\nu = 1$

$\Rightarrow$ $D(S'; b_0', b_1')$ is $(k-p)$-connected, as desired

(Similar casework when $\nu = 2$ or $p \leq 1$)